Washington Latin

AP Calculus

Product Rule

We want the derivative of the product of two functions, each of which we already know how to differentiate. We use the limit of the difference quotient, which is the fundamental definition of the derivative:

$$y = f(x) \cdot g(x)$$

$$y' = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

subtract and add *f(x) · g(x + h)*:

$$y' = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}$$

group:

$$y' = \lim_{h \to 0} \frac{[f(x+h)g(x+h) - f(x)g(x+h)] + [f(x)g(x+h) - f(x)g(x)]}{h}$$

factor out *g(x + h)* and *f(x)* from the from the respective groups:

$$y' = \lim_{h \to 0} \frac{[f(x+h) - f(x)]g(x+h) + f(x)[g(x+h) - g(x)]}{h}$$

break up the sum:

$$y' = \lim_{h \to 0} \frac{[f(x+h) - f(x)]g(x+h)}{h} + \frac{f(x)[g(x+h) - g(x)]}{h}$$

rearrange a little:

$$y' = \lim_{h \to 0} \frac{[f(x+h) - f(x)]}{h} g(x+h) + f(x) \frac{[g(x+h) - g(x)]}{h}$$

and now take the limit, by having *h* go to zero.
The two rational expressions are the derivatives of *f* and *g* respectively. We know:

$$f'(x) = \lim_{h \to 0} \frac{[f(x+h) - f(x)]}{h}$$

and

$$g'(x) = \lim_{h \to 0} \frac{[g(x+h) - g(x)]}{h}$$

so,

$$y' = f'(x)g(x) + f(x)g'(x)$$

Quotient Rule

We want the derivative of the ratio of two functions, each of which we already know how to differentiate. Again, we use the limit of the difference quotient, which is the fundamental definition of the derivative:

$$y = \frac{f(x)}{g(x)}$$

$$y' = \lim_{h \to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h}$$

$$y' = \lim_{h \to 0} \frac{\frac{f(x+h)g(x) - f(x)g(x+h)}{g(x+h)g(x)}}{h}$$

add and subtract *f(x) · g(x):*

$$y' = \lim_{h\to 0} \frac{\frac{f(x+h)g(x) - f(x)g(x) - f(x)g(x+h) + f(x)g(x)}{g(x+h)g(x)}}{h}$$

group terms and simplify by multiplying the numerator by *1/h:*

$$y' = \lim_{h\to 0} \frac{[f(x+h)g(x) - f(x)g(x)] - [f(x)g(x+h) - f(x)g(x)]}{g(x+h)g(x)\cdot h}$$

factor out *f(x)* and *g(x)*

$$y' = \lim_{h\to 0} \frac{g(x)[f(x+h) - f(x)] - f(x)[g(x+h) - g(x)]}{g(x+h)g(x)\cdot h}$$

break up the difference:

$$y' = \lim_{h\to 0} (\frac{g(x)[f(x+h) - f(x)]}{g(x+h)g(x)\cdot h} - \frac{f(x)[g(x+h) - g(x)]}{g(x+h)g(x)\cdot h})$$

re-arrange a little bit:

$$y' = \lim_{h\to 0} (\frac{g(x)}{g(x+h)g(x)} \cdot \frac{f(x+h) - f(x)}{h} - \frac{f(x)}{g(x+h)g(x)} \cdot \frac{g(x+h) - g(x)}{h})$$

and now take the limit, by having *h* go to zero.
The two rational expressions are the derivatives of *f* and *g* respectively. We know:

$$f'(x) = \lim_{h\to 0} \frac{[f(x+h) - f(x)]}{h}$$

and

$$g'(x) = \lim_{h\to 0} \frac{[g(x+h) - g(x)]}{h}$$

$$y' = \frac{g(x)}{g(x)g(x)} \cdot f'(x) - \frac{f(x)}{g(x)g(x)} g'(x)$$

recognizing the common denominator, combine the fractions:

$$y' = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)g(x)}$$

which is Low D-high minus high D-low over low squared.